THE OPEN UNIVERSITY
Mathematics: A Third Level Course
M332 COMPLEX ANALYSIS

M332
Exercise Booklet

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1.1

This booklet consists of problems and their detailed solutions. These solutions generally involve only the use of basic techniques so that you may well find the questions a useful starting point for examination revision. Furthermore the problems may assist you during the course both as a supplement to the units and as a guide to possible strategies for solving some of the continuous assessment questions.

Unit 1

Exercise 1

You are given that $w = 1 + i\sqrt{3}$.

(i) Find $|w|$ and $\operatorname{Arg} w$. Write down $|\overline{w}|$ and $\operatorname{Arg} \overline{w}$.

(ii) If $2z^3 = w$, show that the solution set for z is

$$\left\{ z : |z| = 1 \text{ and } \arg z = \frac{\pi}{9} + \frac{2k\pi}{3} \text{ for } k = 0, 1, 2 \right\}.$$

(iii) Show that the solution set of the equation

$$z^6 - z^3 + 1 = 0$$

can be written as

$$\left\{ z : |z| = 1 \text{ and } \arg z = \pm\frac{\pi}{9} + \frac{2k\pi}{3} \text{ for } k = 0, 1, 2 \right\}.$$

Exercise 2

(i) Given z_1, z_2 are fixed complex numbers and k is a fixed positive real number, describe geometrically each of the following sets of complex numbers:

(a) $\{z : |z - z_1| = |z - z_2|\}$;

(b) $\{z : |z - z_1| = k\}$.

(ii) Sketch the set

$\{z : |z + 1 + i| \leq |z - 1 - i| \leq 2\}$.

Unit 2

Exercise 3

Let f be the function defined by $f(z) = z^4$.

(i) Find the image under f of the set $\{z : 0 \leq \operatorname{Arg} z < \pi/4\}$.

(ii) Find the inverse image under f of the quadrant $\{w : \pi/2 < \operatorname{Arg} w < \pi\}$.

Draw diagrams to illustrate your answers, showing clearly which boundaries are included and which are excluded.

Unit 3

Exercise 4

Evaluate

(i) $\exp(2 + i\,\pi/3)$;

(ii) $\cosh(2 + i\,\pi/3)$;

(iii) $\operatorname{Log}(2i)$;

(iv) principal value of $(2i)^{1/2}$.

Exercise 5

Determine the domain of $\mathrm{Log}(2+iz)$ giving your answer both in set notation and as a clear sketch. Hence deduce the domain of $\sqrt{2+iz}$ where $\sqrt{}$ denotes the principal branch.

Exercise 6

Given $f(z) = \mathrm{Log}(4iz)$, find the inverse of f specifying both its rule and domain.

Exercise 7

Show that

$$f : x+iy \longrightarrow x^2 - y^2 + 3y + 2ixy$$

is not differentiable anywhere.

Unit 4

Exercise 8

(i) Sketch the paths of the arcs γ_1 and γ_2 defined by:

$$\gamma_1(t) = t^2 + 1 + i(2-t^2), \quad t \in [0, \sqrt{2}];$$
$$\gamma_2(t) = 2e^{3it} + 5i, \quad t \in [0, 2\pi].$$

(ii) For each of the arcs γ_1 and γ_2 write down whether or not it is

(a) simple,

(b) closed,

(c) simple-closed,

(d) smooth.

Exercise 9

Evaluate each of the following integrals:

(i) $\int_\Gamma \sec^2 z \exp(\tan z) dz$, where Γ is the lower semi-circle

$$\{z : |z - \pi/8| = \pi/8, \mathrm{Im}\, z \leq 0\} \text{ from } 0 \text{ to } \pi/4;$$

(ii) $\int_C \overline{z}\, dz$, where C is the line segment from 0 to $1+i$.

Exercise 10

Let C be the circle $\{z : |z| = 4\}$. Show that

$$|\sinh z| \leq e^4 \text{ on } C.$$

Hence prove that

$$\left| \int_C \frac{\sinh z}{z^2+1}\, dz \right| \leq \frac{8\pi e^4}{15}.$$

Unit 5

Exercise 11

Given that $C = \{z : |z| = 5\}$, evaluate the following integrals:

(i) $\displaystyle\int_C \frac{\sin z}{z - \pi e}\, dz$

(ii) $\displaystyle\int_C \frac{\sin z}{2z - \pi}\, dz$

(iii) $\displaystyle\int_C \frac{\sin z}{(2z-\pi)^2}\, dz$

Unit 6

Exercise 12

Given $f(z) = \dfrac{z}{z-4}$, $z \neq 4$, find the Taylor series for f about $z = 1$, giving the first 3 non-zero terms and a general term. Write down the radius of convergence of your series.

Unit 8

Exercise 13

For each of the following functions write down:

(i) its singularities;

(ii) whether these singularities are removable, essential or poles;

(iii) the order of any poles

(a) $f(z) = \dfrac{\sin z}{(z-\pi)(2z-\pi)}$

(b) $g(z) = \dfrac{\exp z}{z(z-1)^2 \cos z}$

(c) $h(z) = (z-1)^2 \sin\left(\dfrac{1}{z-1}\right)$

Exercise 14

Let f be the function defined by $f(z) = \dfrac{5z-1}{(z-3)(z+4)}$.

(i) Find a series expansion for f convergent on the annulus $\{z : 3 < |z| < 4\}$.

(ii) Find a series expansion for f convergent on the region $\{z : |z| > 4\}$.

In each case general terms of the expansion should be given.

Unit 9

Exercise 15

It is given that $f(z) = \dfrac{1}{4z^2+1}$ for $|z| < 1/2$.

(i) Show that $\left|f\left(\frac{1}{4}e^{i\theta}\right)\right| = \dfrac{4}{\sqrt{17+8\cos 2\theta}}$.

(ii) Find the maximum value of $|f|$ on the disc $\{z : |z| \leq 1/4\}$, and determine the points z corresponding to this maximum.

Exercise 16

Evaluate $\displaystyle\int_C \frac{\cosh \pi z}{z(z^2+9)}\, dz$ where C is the circle with centre the origin and radius 4.

Unit 10

Exercise 17

For each of the following functions f and points α determine the residue of f at α.

(i) $f(z) = \dfrac{z}{z^3-1}$, $\quad \alpha = 1$.

(ii) $f(z) = \dfrac{\exp z}{z(z-1)^2}$, $\quad \alpha = 1$.

(iii) $f(z) = \dfrac{\sin z}{(z-\pi)^2}$, $\quad \alpha = \pi$.

(iv) $f(z) = (z-4)\cos\left(\dfrac{1}{z-4}\right), \quad \alpha = 4.$

(v) $f(z) = \dfrac{\sinh z}{z}, \quad \alpha = 0.$

Exercise 18

Find a function f such that

$$\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta} = \int_C f(z)\, dz$$

where $C = \{z : |z| = 1\}$, and hence evaluate the first integral.

Exercise 19

Prove the following results:

(i) $\displaystyle\int_0^{\infty} \frac{t^4}{1+t^6}\, dt = \frac{\pi}{3};$

(ii) $\displaystyle\int_0^{\infty} \frac{\sin(kt)}{t(1+t^2)}\, dt = \frac{\pi}{2}\left(1-e^{-k}\right)$, where $k \geq 0$.

Unit 11

Exercise 20

Use Rouché's Theorem to find the number of roots of the equation

$$2z^6 + z^5 + 8z^2 + 4 = 0$$

in

(i) $\{z : |z| < 1\}$;

(ii) $\{z : |z| = 1\}$;

(iii) $\{z : 1 < |z| < 2\}$;

(iv) $\{z : |z| \geq 2\}$.

Unit 13

Exercise 21

Let $S = \{z : |z| < 1, \operatorname{Re} z > 0\}$

(i) Find the image of S under $z \longrightarrow \dfrac{1+iz}{1-iz}$.

(ii) What is the image of $\{z : \operatorname{Re} z > 0, \operatorname{Im} z > 0\}$ under $z \longrightarrow z^2$?

(iii) Determine the image of $\{z : \operatorname{Im} z > 0\}$ under $z \longrightarrow \dfrac{z-i}{z+i}$.

(iv) Using the results of (i), (ii) and (iii) deduce that $w = -i\left(\dfrac{z^2+2z-1}{z^2-2z-1}\right)$ maps S in the z plane onto $T = \{w : |w| < 1\}$ in the w plane.

Unit 14

Exercise 22

(i) The real function f satisfies

$$f''(t) - 5f'(t) + 6f(t) = 12$$

where $f(0) = f'(0) = 0$.
Determine the Laplace transform of f.

(ii) Verify that you may use the Inversion Formula to find f and hence show that

$$f(t) = 2 - 6e^{2t} + 4e^{3t}.$$

Solutions

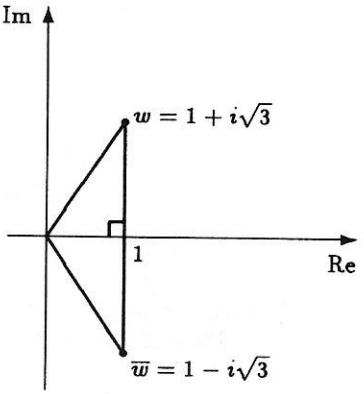
There are often alternative methods of arriving at the solutions to the examples in this booklet. The ones given below are those followed by one particular tutor. You may prefer a different approach. However, the methods used here are straightforward and apply to a wide variety of problems in Complex Analysis.

Solution 1

(i) A rough sketch is useful to ensure correct quadrant for $\operatorname{Arg} w$

$|w| = 2; \quad \operatorname{Arg} w = \pi/3.$

$|\overline{w}| = 2; \quad \operatorname{Arg} \overline{w} = -\pi/3.$



(ii) $$z^3 = \frac{1}{2}\left(1 + i\sqrt{3}\right)$$

$$= \cos\left(\frac{\pi}{3} + 2k\pi\right) + i \sin\left(\frac{\pi}{3} + 2k\pi\right)$$

$$z = \cos\left(\frac{\pi}{9} + \frac{2k\pi}{3}\right) + i \sin\left(\frac{\pi}{9} + \frac{2k\pi}{3}\right), \text{ for } k = 0, 1, 2.$$

Hence solution set is $\left\{z : |z| = 1 \text{ and } \arg z = \frac{\pi}{9} + \frac{2k\pi}{3} \text{ for } k = 0, 1, 2\right\}$.

(iii) $$z^6 - z^3 + 1 = 0$$

Solving this quadratic in z^3 gives

$$z^3 = \frac{1}{2}\left(1 \pm i\sqrt{3}\right).$$

From (ii) solution set for $z^3 = \frac{1}{2}\left(1 + i\sqrt{3}\right)$ is

$$\left\{z : |z| = 1 \text{ and } \arg z = \frac{\pi}{9} + \frac{2k\pi}{3} \text{ for } k = 0, 1, 2\right\}.$$

Similarly solution set for $z^3 = \frac{1}{2}\left(1 - i\sqrt{3}\right)$ is

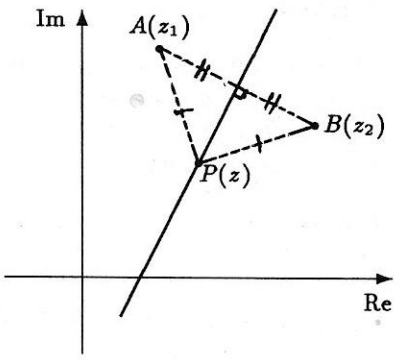
$$\left\{z : |z| = 1 \text{ and } \arg z = -\frac{\pi}{9} + \frac{2k\pi}{3} \text{ for } k = 0, 1, 2\right\}.$$

Thus required solution set is

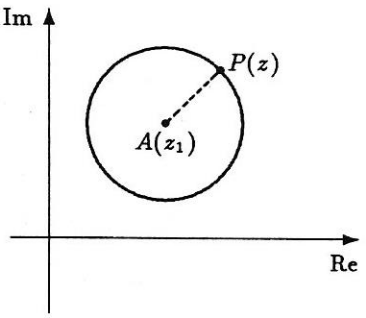
$$\left\{z : |z| = 1 \text{ and } \arg z = \pm\frac{\pi}{9} + \frac{2k\pi}{3} \text{ for } k = 0, 1, 2\right\}.$$

Solution 2

(i) (a) $|z - z_1| = |z - z_2| \iff |PA| = |PB|$.
Thus the set is the perpendicular bisector of the line joining z_1 to z_2.

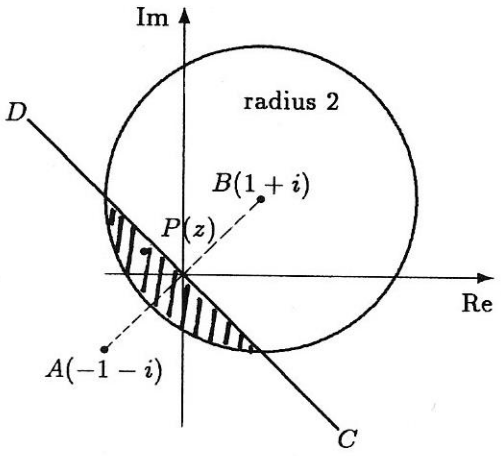


(b) $|z - z_1| = k \iff |AP| = k$.
Thus the set is the circle with centre z_1 and radius k.



(ii) $|z + 1 + i| \leq |z - 1 - i|$ consists of line CD together with all points below CD, the perpendicular bisector of AB.

$|z - 1 - i| \leq 2$ consists of the circle shown together with its interior. Hence the given set is the shaded segment including the boundary of the segment.



Solution 3

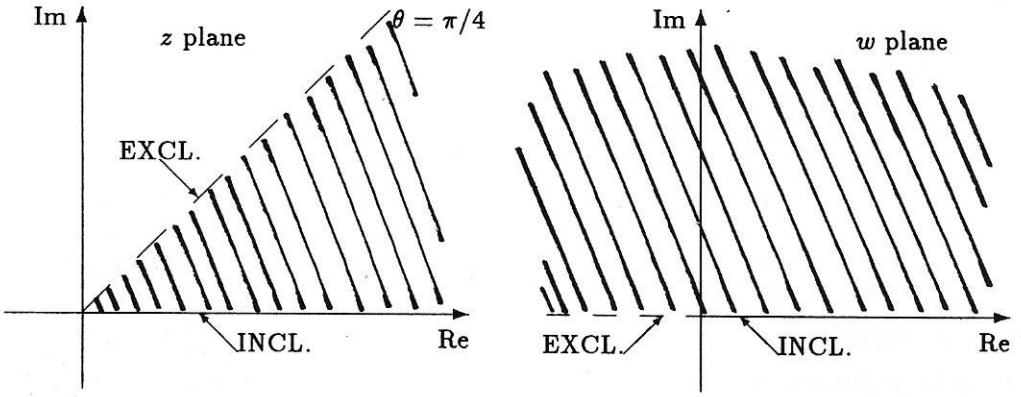
Let $w = z^4$. Thus $\rho e^{i\phi} = \left(re^{i\theta}\right)^4$

$\iff \rho e^{i(\phi+2k\pi)} = r^4 e^{4i\theta}$ where $k = 0, \pm 1, \pm 2, \ldots$

$\iff \rho = r^4$ and $\phi + 2k\pi = 4\theta$.

$\rho \geq 0 \iff r \geq 0$ (remember ρ and r cannot be negative).

(i) $\left\{z : 0 \le \theta < \frac{\pi}{4}\right\} = \{z : 0 \le 4\theta < \pi\} \longrightarrow \{w : 0 \le \phi + 2k\pi < \pi\} = \{w : 0 \le \phi < \pi\}.$

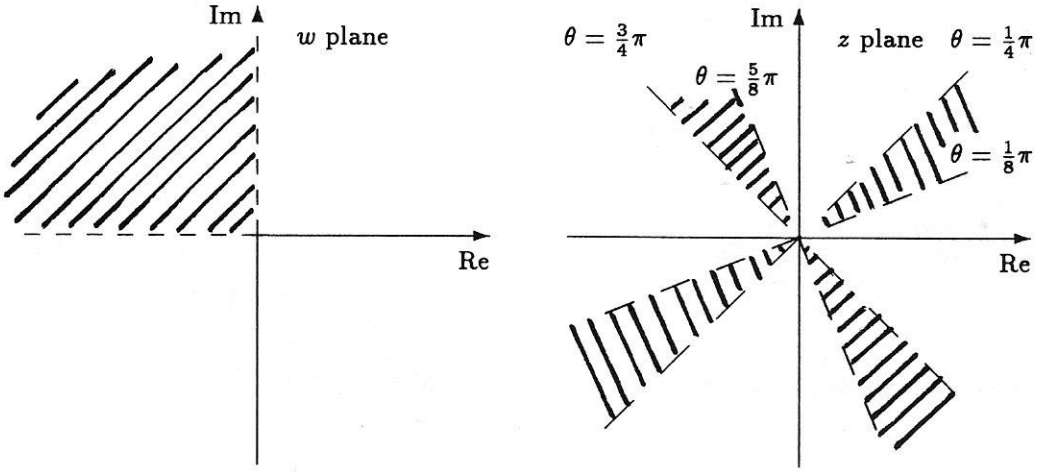


(ii) $w \in \{w : \pi/2 < \operatorname{Arg} w < \pi\} \iff z \in \{z : \pi/2 < 4\theta - 2k\pi < \pi\}$, for $k = 0, \pm 1, \pm 2, \ldots$

$$\iff z \in \left\{z : (4k+1)\frac{\pi}{8} < \theta < (2k+1)\frac{\pi}{4}\right\}, \text{ for } k = 0, \pm 1, \pm 2, \ldots.$$

so the inverse image is

$$\left\{z : \frac{\pi}{8} < \theta < \frac{\pi}{4}\right\} \cup \left\{z : \frac{5\pi}{8} < \theta < \frac{3\pi}{4}\right\} \cup \left\{z : \frac{9\pi}{8} < \theta < \frac{5\pi}{4}\right\} \cup \left\{z : \frac{13\pi}{8} < \theta < \frac{7\pi}{4}\right\}.$$



Solution 4

(i) $\exp(2 + i\pi/3) = e^2(\cos\ \pi/3 + i \sin\ \pi/3)$

$$= \frac{1}{2}e^2\left(1 + i\sqrt{3}\right).$$

(ii) $\exp[-(2 + i\pi/3)] = e^{-2}[\cos(-\pi/3) + i\sin(-\pi/3)]$

$$= \frac{1}{2e^2}\left(1 - i\sqrt{3}\right).$$

Hence $\cosh(2 + i\pi/3) = \frac{1}{4}\left[\left(e^2 + \frac{1}{e^2}\right) + i\sqrt{3}\left(e^2 - \frac{1}{e^2}\right)\right]$

$$= \frac{1}{2}\left(\cosh 2 + i\sqrt{3}\sinh 2\right).$$

(iii) $2i = 2\exp(i\pi/2)$.
Thus $\text{Log}(2i) = (\log 2) + i\pi/2$.

(iv) p.v. of $(2i)^{\frac{1}{2}} = \exp\left[\frac{1}{2}\text{Log}(2i)\right]$
$= \exp\left[\log(\sqrt{2}) + i\pi/4\right]$
$= \sqrt{2}(\cos\ \pi/4 + i\sin\ \pi/4)$
$= 1 + i.$

Solution 5

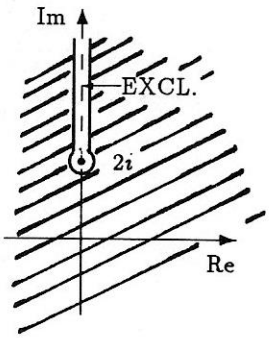
$2 + iz \neq -k$ where $k \geq 0 \iff z \neq (2+k)i,\ k \geq 0.$

Thus domain of $\text{Log}(2+iz)$ is

$\{z : z \in \mathbf{C} \text{ and } z \neq (2+k)i,\ k \geq 0\}.$

Now the principal branch of $\sqrt{2+iz}$ is $\exp\left[\frac{1}{2}\text{Log}(2+iz)\right]$.
Hence domain of principal branch of $\sqrt{2+iz}$ is

$\{z : z \in \mathbf{C} \text{ and } z \neq (2+k)i,\ k \geq 0\}.$



Solution 6

Let $w = \text{Log}(4iz)$.
$\exp w = \exp \text{Log}(4iz) = 4iz.$
Hence $z = -\frac{1}{4}\, i\, \exp w.$

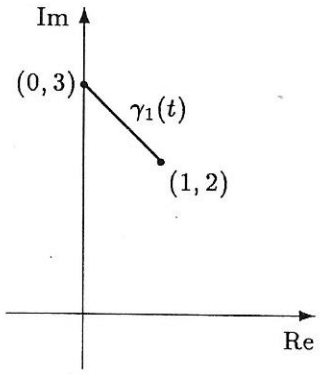
Inverse rule is $z \longmapsto -\frac{1}{4} i \exp z$.
Inverse domain is $\{z : z = x + iy \text{ where } -\pi < y < \pi\}$.

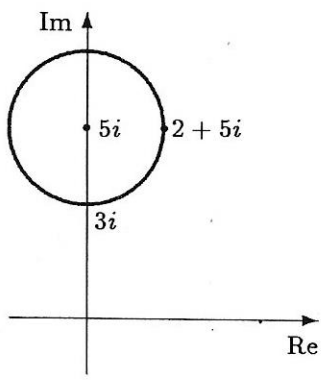
Solution 7

Let $f = U + iV$ so that $U = x^2 - y^2 + 3y$ and $V = 2xy$.
$\frac{\partial U}{\partial x} = 2x = \frac{\partial V}{\partial y}$ but $\frac{\partial U}{\partial y} = -2y + 3$ and $\frac{\partial V}{\partial x} = 2y.$
Hence $\frac{\partial U}{\partial y}$ is *never* equal to $-\frac{\partial V}{\partial x}$.
Thus Cauchy-Riemann equations are not satisfied at any point z showing f is not differentiable anywhere.

Solution 8

(i) $\gamma_1(t) = x + iy$ where $x = t^2 + 1$ and $y = -t^2 + 2$. Thus $x + y = 3$ and the path starts at (1,2) and finishes at (3,0). $\gamma_2(t) = x + iy$ where $x = 2\cos 3t$ and $y = 5 + 2\sin 3t$. Hence $x^2 + (y-5)^2 = 4$ which is a circle with centre (0,5) and radius 2. Period of $\cos 3t$ is $2\pi/3$. Now t varies from 0 to 2π so that circle is described three times with $\gamma_2(0) = 2 + 5i$





(ii) γ_1 is simple (one-one), not closed nor simple-closed and is not smooth as $\gamma'(0) = 0$. (See definition on page 21 of *Unit 4*.)

γ_2 is not simple (as eg. $\gamma_2(0) = \gamma_2(2\pi/3)$) and hence not simple-closed. It is closed as $\gamma_2(2\pi) = \gamma_2(0)$. $\gamma_2'(t) = 6ie^{3it}$ is never zero and as $\gamma_2'(2\pi) = \gamma_2'(0) = 6i$, γ_2 is smooth. (See the top of page 22 of *Unit 4*.)

Solution 9

(i) Let $f'(z) = \sec^2 z \exp(\tan z)$ so that we can take $f(z) = \exp(\tan z)$.
f is analytic on $R = \{z : |z - \pi/8| < \pi/4\}$.
Γ is a contour in R and f' is continuous in R.
Hence we can apply "The Fundamental Theorem for Contour Integrals" to obtain

$$\begin{aligned}\int_\Gamma \sec^2 z \exp(\tan z) dz &= \exp(\tan \pi/4) - \exp(\tan 0) \\ &= e - 1.\end{aligned}$$

(ii) Since $\overline{z}$ is not the derivative of an analytic function we cannot use the method of the solution to part (i). Instead we shall use the x, y-equation of the line segment.
On C we have $y = x$ so that $\overline{z} = x - iy = (1 - i)x$ and
$dz = d(x + iy) = d(x + ix) = (1 + i)dx$.
Thus $\int_C \overline{z}\, dz = (1 - i^2) \int_0^1 x\, dx = 1$.

Solution 10

$$\begin{aligned}|\sinh z| &= \frac{1}{2}\left|e^{x+iy} - e^{-x-iy}\right| \\ &\leq \frac{1}{2}\left(\left|e^{x+iy}\right| + \left|e^{-x-iy}\right|\right), \text{ using the triangle inequality,} \\ &= \frac{1}{2}\left(e^x + e^{-x}\right) \\ &\leq \frac{1}{2}\left(e^4 + e^4\right) \text{ on } C \\ &= e^4.\end{aligned}$$

$f(z) = \dfrac{\sinh z}{z^2 + 1}$ is analytic except at $z = \pm i$. Hence f is continuous on C. Hence we can apply "The Estimation Theorem". Now $L = 8\pi$ and $|z^2 + 1| \geq |z^2| - 1 = 15$ on $|z| = 4$.
Thus $\left|\displaystyle\int_C \frac{\sinh z}{z^2 + 1}\, dz\right| \leq \left(\frac{e^4}{15}\right) 8\pi = \frac{8\pi e^4}{15}$.

Solution 11

(i) $\dfrac{\sin z}{z - \pi e}$ is analytic except at $z = \pi e$ and this point lies outside $|z| = 5$. Hence using Cauchy's Theorem for Star Regions we have

$$\int_C \frac{\sin z}{z - \pi e}\, dz = 0.$$

(ii) $\frac{1}{2}\sin z$ is entire. Also $z = \frac{\pi}{2}$ is inside C. Hence Cauchy's Formula gives

$$\int_C \frac{\frac{1}{2}\sin z}{z - \frac{\pi}{2}}\, dz = 2\pi i \left(\frac{1}{2}\sin\frac{\pi}{2}\right).$$

Thus $\displaystyle\int_C \frac{\sin z}{2z - \pi}\, dz = \pi i.$

(iii) As in (ii) but using Cauchy's Formula for the First Derivative we get

$$\int_C \frac{\frac{1}{4}\sin z}{(z - \frac{\pi}{2})^2}\, dz = 2\pi i \left(\frac{1}{4}\cos\frac{\pi}{2}\right) = 0.$$

Thus $\displaystyle\int_C \frac{\sin z}{(2z - \pi)^2}\, dz = 0.$

Solution 12

Let $u = z - 1$ to expand about $u = 0$.

$$\begin{aligned} f(z) &= \frac{1+u}{-3+u} \\ &= -\frac{1}{3}(1+u)\left(1-\frac{u}{3}\right)^{-1} \\ &= -\frac{1}{3}(1+u)\left[1+\frac{u}{3}+\left(\frac{u}{3}\right)^2+\cdots+\left(\frac{u}{3}\right)^n+\cdots\right] \\ &= -\frac{1}{3}\left(1+\frac{4u}{3}+\frac{4u^2}{9}+\cdots+\frac{4u^n}{3^n}+\cdots\right) \\ &= -\frac{1}{3}-\frac{4}{3^2}(z-1)-\frac{4}{3^3}(z-1)^2-\cdots-\frac{4}{3^{n+1}}(z-1)^n-\cdots. \end{aligned}$$

Radius of convergence is 3. [Later on in the course this result is obtained quickly using distance from "centre" of Taylor expansion (in our case $z = 1$) to nearest point at which f is non–analytic (in our case $z = 4$).]

Solution 13

	(i)	(ii)	(iii)
(a)	$\frac{\pi}{2}$	pole	1
	π	removable[1]	–
(b)	0	pole	1
	1	pole	2
	$(2k+1)\frac{\pi}{2}$, $k \in \mathbf{Z}$	pole	1
(c)	1	essential[2]	–

Notes

1 For, a removable singularity for f at α it is sufficient to show $\lim_{z\to\alpha} f(z)$ exists. Use L'Hôpital's Rule. (See foot of page 14 and the top of page 15 of *Unit 8*.)

2 Let $f(z) = \sum_{n=0}^{\infty} a_n(z-\alpha)^n + \sum_{n=1}^{\infty} b_n(z-\alpha)^{-n}$.

If $\{b_n\}$ does not consist of a finite number of b's, then the singularity at α is essential.

Solution 14

$$f(z) = \frac{2}{z-3} + \frac{3}{z+4}.$$

(i)
$$\begin{aligned} f(z) &= \frac{2}{z(1-\frac{3}{z})} + \frac{3}{4(1+\frac{z}{4})} \\ &= \frac{2}{z}\left(1+\frac{3}{z}+\frac{3^2}{z^2}+\cdots+\frac{3^n}{z^n}+\cdots\right)+\frac{3}{4}\left(1-\frac{z}{4}+\frac{z^2}{4^2}-\cdots+\frac{(-1)^n z^n}{4^n}+\cdots\right) \end{aligned}$$

where $|\frac{3}{z}| < 1$ *and* $|\frac{z}{4}| < 1$ i.e. on the annulus $\{z : 3 < |z| < 4\}$.

(ii)
$$\begin{aligned} f(z) &= \frac{2}{z(1-\frac{3}{z})} + \frac{3}{z(1+\frac{4}{z})} = \frac{2}{z}\left(1+\frac{3}{z}+\frac{3^2}{z^2}+\cdots+\frac{3^n}{z^n}+\cdots\right)+\frac{3}{z}\left(1-\frac{4}{z}+\frac{4^2}{z^2}-\cdots+(-1)^n\frac{4^n}{z^n}+\cdots\right) \\ &= \frac{5}{z}-\frac{6}{z^2}+\frac{66}{z^3}-\cdots+\frac{6}{z^{n+1}}\left(3^{n-1}+(-1)^n 2^{2n-1}\right)+\cdots, \text{ where } |z| > 4. \end{aligned}$$

Solution 15

(i)
$$\left|f\left(\frac{1}{4}e^{i\theta}\right)\right| = \frac{1}{|\frac{1}{4}e^{2i\theta}+1|}$$
$$= \frac{4}{|4+\cos 2\theta + i\sin 2\theta|}$$
$$= \frac{4}{\sqrt{(4+\cos 2\theta)^2+\sin^2 2\theta}}$$
$$= \frac{4}{\sqrt{17+8\cos 2\theta}}.$$

(ii) Let $R = \{z : |z| < 1/4\}$
Then f is analytic on the bounded region R. Further f is continuous on $\overline{R}$. Hence we can use the Maximum Principle to deduce that $|f|$ has a max on $\{z : |z| \leq 1/4\}$ at a point on $|z| = 1/4$ i.e. at $z = \frac{1}{4}e^{i\theta}$. Hence part (i) gives $|f|_{\max} = \frac{4}{\sqrt{17-8}} = \frac{4}{3}$ occurring when $\cos 2\theta = -1$ i.e. when $\theta = \frac{1}{2}(2k+1)\pi$ for $k = 0, \pm 1, \pm 2, \ldots$.
Hence $z = \pm\, i/4$ give a maximum of $4/3$ for $|f|$.

Solution 16

Let $f(z) = \dfrac{\cosh \pi z}{z(z^2+9)} = \dfrac{\cosh \pi z}{z(z-3i)(z+3i)}$.
Poles of f occur at $z = 0$, $3i$, $-3i$ and each of these simple poles lies inside C. Using the "Cover-up Rule" we have

$$\text{Res}(f, 0) = \frac{\cosh 0}{0+9} = \frac{1}{9}.$$
$$\text{Res}(f, 3i) = \frac{\cosh(3\pi i)}{(3i)(3i+3i)} = \frac{\cos(3\pi)}{-18} = \frac{1}{18}.$$
$$\text{Res}(f, -3i) = \frac{\cosh(-3\pi i)}{(-3i)(-6i)} = \frac{1}{18}.$$

The Residue Theorem now gives

$$\int_C \frac{\cosh \pi z}{z(z^2+9)}\, dz = 2\pi i\left(\frac{1}{9}+\frac{1}{18}+\frac{1}{18}\right) = \frac{4\pi i}{9}.$$

Solution 17

(i) $\text{Res}(f, 1) = \left[\dfrac{z}{3z^2}\right]_{z=1} = \dfrac{1}{3}$.

(ii) $\text{Res}(f, 1) = \left[\dfrac{d}{dz}\left(\dfrac{\exp z}{z}\right)\right]_{z=1} = 0.$

(iii) $z = \pi$ is a simple pole. Hence
$\text{Res}(f, \pi) = \lim\limits_{z\to\pi} \dfrac{\sin z}{(z-\pi)} = \lim\limits_{z\to\pi} \dfrac{\cos z}{1} = -1$, by L'Hôpital.

(iv)
$$f(z) = (z-4)\left[1 - \frac{1}{2(z-4)^2} + \ldots\right]$$
$$= (z-4) - \left(\frac{1}{2}\right)\left(\frac{1}{z-4}\right) + \ldots.$$

$\text{Res}(f, 4) = -\dfrac{1}{2}$.

(v) $\lim\limits_{z\to 0} \dfrac{\sinh z}{z} = \lim\limits_{z\to 0} \dfrac{\cosh z}{1} = 1$, by L'Hôpital's Rule.
Hence $z = 0$ is removable.
Thus $\text{Res}(f, 0) = 0$.

Solution 18

On C we have $z = e^{i\theta}$ so that $\dfrac{dz}{d\theta} = ie^{i\theta}$. Hence $d\theta = \dfrac{dz}{iz}$ and $\cos\theta = \frac{1}{2}(z + 1/z)$.

Hence $\displaystyle\int_0^{2\pi} \frac{d\theta}{5 + 3\cos\theta} = \int_C \frac{2dz}{i\left[10z + 3\left(z^2 + 1\right)\right]}$

$$\text{giving } f(z) = \frac{2}{(3z^2 + 10z + 3)\,i}$$

$$= \frac{2}{(3z + 1)(z + 3)i}.$$

Pole $z = -\frac{1}{3}$ lies inside C.

$$\text{Res}\left(f, -\frac{1}{3}\right) = \left[\frac{2}{(6z + 10)i}\right]_{z=-\frac{1}{3}}$$

$$= \frac{1}{4i}.$$

[Note, if using the Cover-up Rule you must write $(3z + 1)$ as $3(z + \frac{1}{3})$ first.]
Residue Theorem now gives

$$\int_0^{2\pi} \frac{d\theta}{5 + 3\cos\theta} = 2\pi i\left(\frac{1}{4i}\right) = \frac{\pi}{2}.$$

Solution 19

(i) Let $p(t) = t^4$ and $q(t) = t^6 + 1$

Then the conditions of Theorem 8(a) are satisfied with $\beta = 0$. Since q has no zeros on real axis (this does not affect condition (i) of the Theorem 8(a)) we have $T = 0$. Taking $f(z) = \frac{p(z)}{q(z)}$ the poles of f on the upper half plane occur when $z^6 = -1$ i.e. at $z_k = \cos(2k + 1)\frac{\pi}{6} + i\sin(2k + 1)\frac{\pi}{6}$ for $k = 0$, 1,2.

$$\text{Res}(f, z_k) = \left[\frac{t^4}{6t^5}\right]_{t=z_k} = \frac{1}{6}\left[\cos(2k + 1)\frac{\pi}{6} - i\sin(2k + 1)\frac{\pi}{6}\right]$$

Hence $S = \text{Res}(f, z_0) + \text{Res}(f, z_1) + \text{Res}(f, z_2)$

$$= \frac{1}{6}\left[\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right] + \frac{1}{6}\left[\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right] + \frac{1}{6}\left[\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}\right]$$

$$= -i/3.$$

Thus $\displaystyle\int_{-\infty}^{\infty} \frac{t^4}{1 + t^6}\,dt = \text{Re}(2\pi i S) = \frac{2\pi}{3}.$

Now $\dfrac{t^4}{1 + t^6}$ is an even function. Hence

$$\int_0^{\infty} \frac{t^4}{1 + t^6}\,dt = \frac{\pi}{3}.$$

(ii) Let $p(t) = 1$ and $q(t) = t\left(1 + t^2\right)$. The degree of q exceeds that of p by 3 and q has a simple zero at $t = 0$ on real axis. Further if we take $\beta = k$ with $k > 0$ (the required result clearly holds for $k = 0$) then all the conditions of Theorem 8(b) are satisfied.

$$T = \text{Res}(f, 0) \text{ where } f(z) = \frac{e^{ikz}}{z(z^2 + 1)},$$

$$= 1.$$

$$S = \text{Res}(f, i)$$

$$= \left[\frac{e^{ikz}}{3z^2 + 1}\right]_{z=i}$$

$$= -\frac{1}{2}e^{-k}.$$

Hence $\displaystyle\int_{-\infty}^{\infty} \frac{\sin(kt)}{t(1 + t^2)}\,dt = \text{Im}\left(-\pi i e^{-k} + \pi i\right).$

$$= \pi\left(1 - e^{-k}\right).$$

Result follows because $\dfrac{\sin(kt)}{t(1+t^2)}$ is even.

Solution 20

Let $f(z) = 8z^2 + 4$ and $g(z) = 2z^6 + z^5$.

(i) $|f(z)| \geq |8z^2| - 4 = 4$ on $|z| = 1$;
$|g(z)| \leq |2z^6| + |z^5| = 3$ on $|z| = 1$.

By Rouché's Theorem $f + g$ has same number of zeros inside $|z| = 1$ as f. The roots of $f = 0$ are $\pm i/\sqrt{2}$ and both of these are inside $|z| = 1$. Hence $f + g = 0$ has 2 roots inside $|z| = 1$.

(ii) $f + g = 0 \iff f = -g \implies |f| = |g|$ which from (i) above is false. Hence $f + g = 0$ has no roots on $|z| = 1$.

(iii) $|g(z)| \geq |2z^6| - |z^5| = 96$ on $|z| = 2$;
$|f(z)| \leq |8z^2| + 4 = 36$ on $|z| = 2$.

Thus $f + g = 0$ has same number of roots inside $|z| = 2$ as $g = 0$ has.
Now $2z^6 + z^5 = 0 \implies z = 0$ (5 times) *or* $z = -\frac{1}{2}$.
Hence $f + g = 0$ has 6 roots inside $|z| = 2$.
We deduce that there are 4 roots in $\{z : 1 < |z| < 2\}$.

(iv) $f + g = 0$ has only 6 roots and all of these are inside $|z| = 2$. Hence $f + g = 0$ has no roots in $\{z : |z| \geq 2\}$.

Solution 21

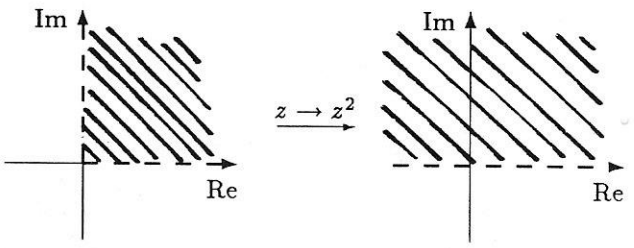
(i) $$w = \frac{1+iz}{1-iz} \iff iz = \frac{w-1}{w+1}.$$

Hence $|z| < 1 \iff |w-1| < |w+1| \iff \operatorname{Re} w > 0$.

$$\begin{aligned}
\text{Also } \operatorname{Re} z > 0 &\iff \operatorname{Im}(iz) > 0 \\
&\iff \operatorname{Im}\left(\frac{w-1}{w+1}\right) > 0 \\
&\iff \operatorname{Im}\left[(w-1)(\overline{w}+1)\right] > 0 \text{ as } |w+1|^2 > 0, \\
&\iff \operatorname{Im}(w - \overline{w}) > 0 \\
&\iff \operatorname{Im} w > 0.
\end{aligned}$$

Thus image of S is the quadrant $\{w : \operatorname{Re} w > 0, \operatorname{Im} w > 0\}$.
[See Example 3 on page 63 for an alternative method.]

(ii) Image of the given quadrant is the upper half plane $\{z : \operatorname{Im} z > 0\}$.



(iii) $$w = \frac{z-i}{z+i} \iff z = -i\left(\frac{w+1}{w-1}\right).$$

$$\begin{aligned}
\operatorname{Im} z > 0 &\iff \operatorname{Re}\left(\frac{w+1}{w-1}\right) < 0 \\
&\iff \operatorname{Re}\left[(w+1)(\overline{w}-1)\right] < 0 \text{ as } |w-1|^2 > 0, \\
&\iff \operatorname{Re}\left[w\overline{w} - 1 - (w - \overline{w})\right] < 0 \\
&\iff \{w : |w| < 1\} \text{ as } w\overline{w} = |w|^2 \text{ and } w - \overline{w} = 2i \operatorname{Im} w.
\end{aligned}$$

Required image is interior of the unit circle.

(iv)
$$w = \frac{\left(\frac{1+iz}{1-iz}\right)^2 - i}{\left(\frac{1+iz}{1-iz}\right)^2 + i}$$
$$= -i\left(\frac{z^2+2z-1}{z^2-2z-1}\right).$$

Solution 22

(i) $\left(p^2 - 5p + 6\right) F(p) = \dfrac{12}{p}$.

So $F(p) = \dfrac{12}{p(p-2)(p-3)}$.

(ii) *First Condition* F is analytic on the region containing the half plane $\operatorname{Re} p \geq 4$.

Second Condition Take $p_0 = 6$. For $|p| > 6$ we have

$$|p-2| \geq |p| - 2 > |p| - 3 > \frac{1}{2}|p|,$$
$$|p-3| \geq |p| - 3 > \frac{1}{2}|p|.$$

Hence $|F(p)| < \dfrac{12}{|p|\frac{1}{2}|p|\frac{1}{2}|p|} = \dfrac{48}{|p|^3}$.

Thus $k = 3$, $K = 48$ and $p_0 = 6$.

Third Condition $F(p)$ is clearly real when p is real.

$$f(t) = \sum \text{ residues of } e^{pt}F(p)$$
$$= \frac{12}{(-2)(-3)} + \frac{12e^{2t}}{(2)(-1)} + \frac{12e^{3t}}{(3)(1)}, \text{ using "Cover-Up" Rule}$$
$$= 2 - 6e^{2t} + 4e^{3t}.$$

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